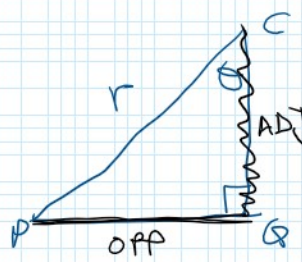
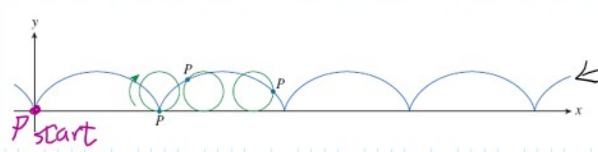
Week 14
Sunday, April 23, 2023 1:52 PM

Leftover from §1.6:

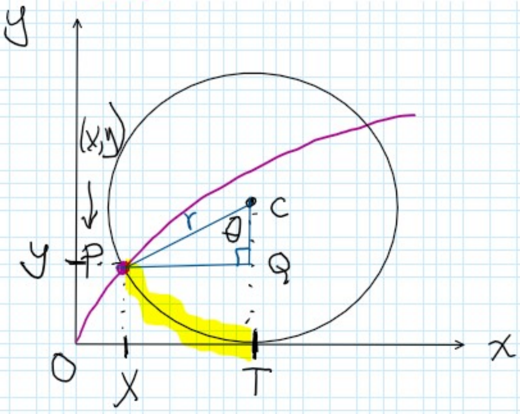
<u>cycloid</u> := curved traced by a point on the circumference of a circle as it rolls along a straight path

for example, if unit circle rolls along the x-axis and if one position of P is <u>at the origin</u>, find a set of parametric equations of the cycloid





← cycloid (path generated by the point P's movement)



$\sin\theta = \frac{|PQ|}{r}$

$\cos\theta = \frac{|QC|}{r}$

$L_{\widehat{PT}} = r\theta$
$\|$
$|OT|$

$|PC| = r$
$|TC| = r$

write x, y ITO r, θ

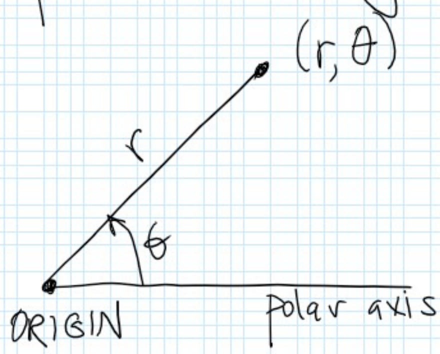
$x = |OT| - |PQ|$ $\quad$ $y = |TC| - |QC|$
$\quad r\theta - r\sin\theta \qquad\quad r - r\cos\theta$

$$\boxed{x = r(\theta - \sin\theta) \quad y = r(1-\cos\theta)}$$

xy-plane uses (x, y) ← Cartesian coordinates represent directed distances from 2 ⊥ axes

Polar Coordinates – Appendix H

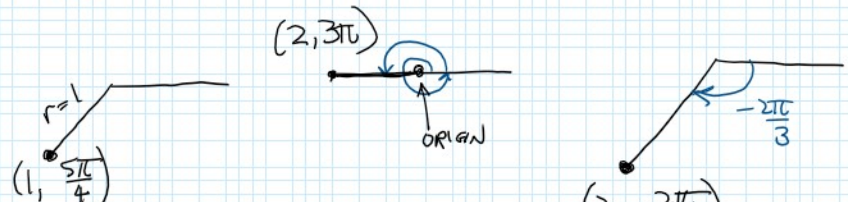
in polar coordinate system, ordered pairs represent:
1. fixed point (pole) radius from r
2. angle of rotation θ



Plot in Polar – assume $r > 0$

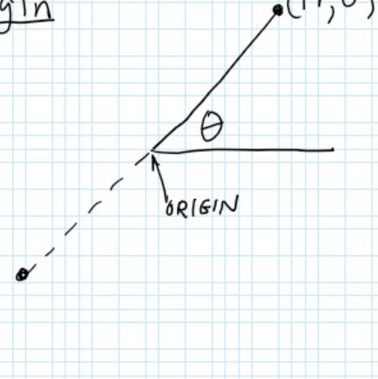
ex. $\left(1, \frac{5\pi}{4}\right)$ $\qquad$ $\left(2, 3\pi\right)$ $\qquad$ $\left(2, -\frac{2\pi}{3} + 2\pi\right)$

1st sketch *



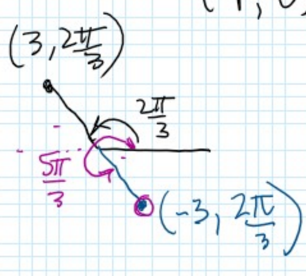
Plot when $r < 0$:

to negate a radius, reflect about <u>origin</u>



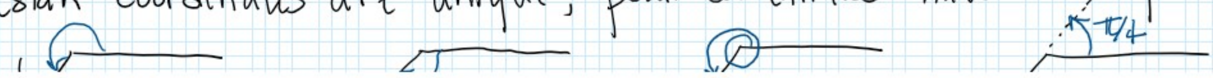
ex. plot $\left(-3, \frac{2\pi}{3}\right)$

1st plot $\left(+3, \frac{2\pi}{3}\right)$



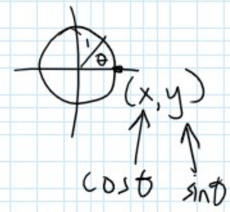
$\left(-3, \frac{2\pi}{3}\right)$ ← same as $\left(+3, \frac{5\pi}{3}\right)$

Cartesian coordinates are unique, polar coordinates have multiple representations:



Cartesian coordinates are unique, polar coordinates have multiple representations.

Convert Between Polar and Cartesian:



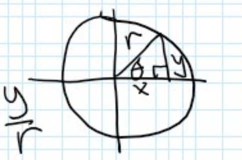
$\boxed{\text{Polar to Cartesian : } x = r\cos\theta \qquad y = r\sin\theta}$

ex- convert $(2, \frac{\pi}{3})$ from polar to Cartesian
r ↑ θ ↑

$x = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$

$y = 2\sin\frac{\pi}{3} = \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} = \sqrt{3}$

$\sin\theta = \frac{y}{r}$

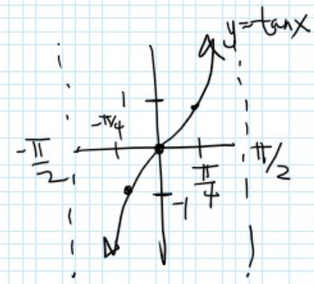


$(1, \sqrt{3})$

$\boxed{\text{Cartesian to Polar: } r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}}$

don't use $\theta = \arctan\frac{y}{x}$
restricted domain

ex. convert $(1,-1)$ from Cartesian to polar
x ↑ y ↑



$r^2 = 1^2 + (-1)^2$
$= 2$
$r = \sqrt{2}$

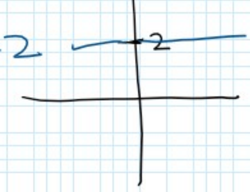
$\tan\theta = \frac{-1}{1} = -1$
could be $-\frac{\pi}{4}$

$(\sqrt{2}, -\frac{\pi}{4})$ or $(\sqrt{2}, \frac{7\pi}{4})$
coterminal ∡'s

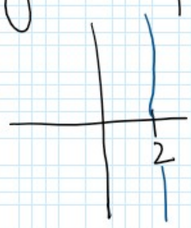
Identifying Curves in Polar

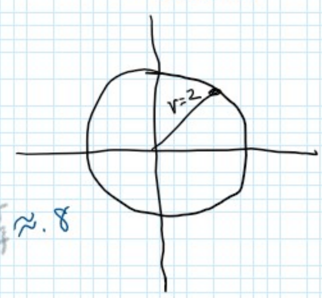
ex. what curve is represented by the polar equation $r = 2$?
(applies to all ∡'s of rotation)

similiar: $y=2$

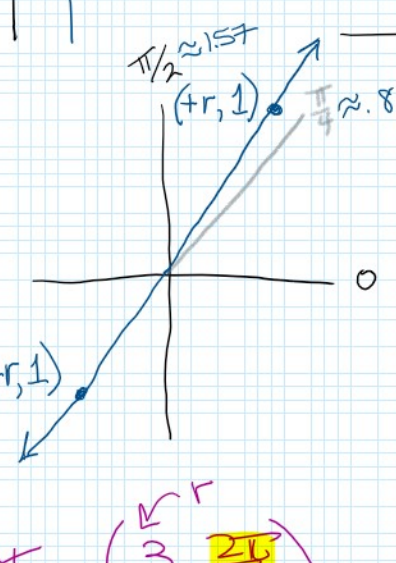


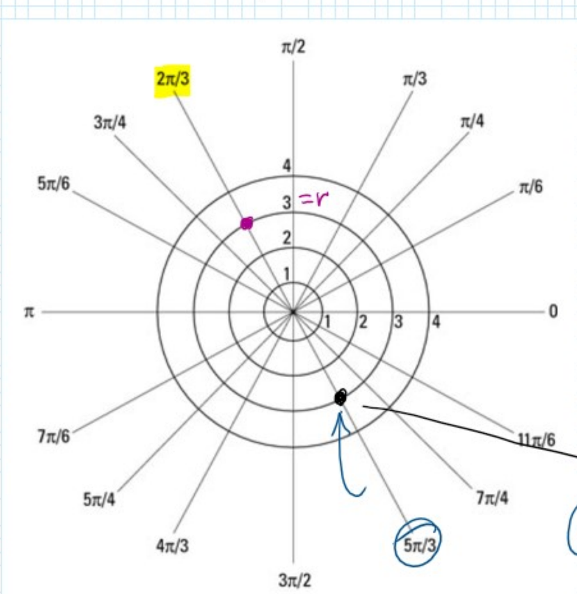
$x=2$





ex. sketch polar curve $\theta = 1$
radians

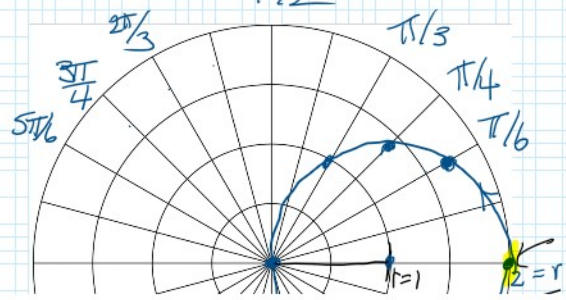




plot $(3, \frac{2\pi}{3})$
r

$(-3, \frac{2\pi}{3})$

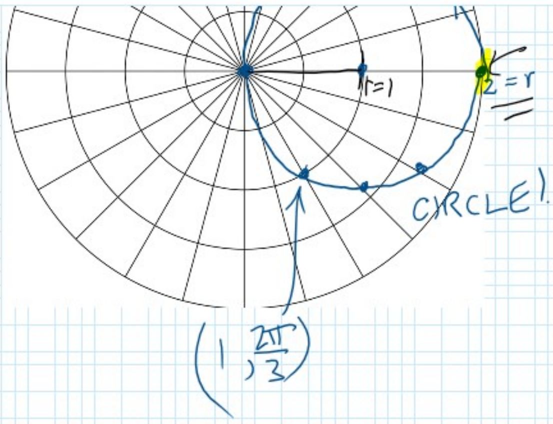
$(+3, \frac{5\pi}{3})$



ex. sketch curve with polar equation $\boxed{r = 2\cos\theta}$

θ	$r = 2\cos\theta$
0	2
$\frac{\pi}{6}$	$\sqrt{3} \approx 1.7$
$\frac{\pi}{4}$	$\sqrt{2} \approx 1.4$

$2\cos\frac{\pi}{6} = \cancel{2}\frac{\sqrt{3}}{\cancel{2}}$



$\frac{\pi}{6}$	
$\frac{\pi}{4}$	$\sqrt{2} \approx 1.4$
$\frac{\pi}{3}$	1 ←
$\frac{\pi}{2}$	0 ←
$\frac{2\pi}{3}$	−1
$\frac{3\pi}{4}$	$-\sqrt{2}$
$\frac{5\pi}{6}$	$-\sqrt{3}$
π	−2

&

use conversion formulas:

$x = r\cos\theta$

$r^2 = x^2 + y^2$

Follow Up Q: find a Cartesian equation for this curve (circle)

$x = r\cos\theta$ $\qquad$ $r = 2\cos\theta$

$\boxed{\frac{x}{r}} = \cos\theta$ $\qquad$ $r = 2 \cdot \frac{x}{r} \Rightarrow r^2 = 2x$

$\therefore\ r^2 = 2x = x^2 + y^2$

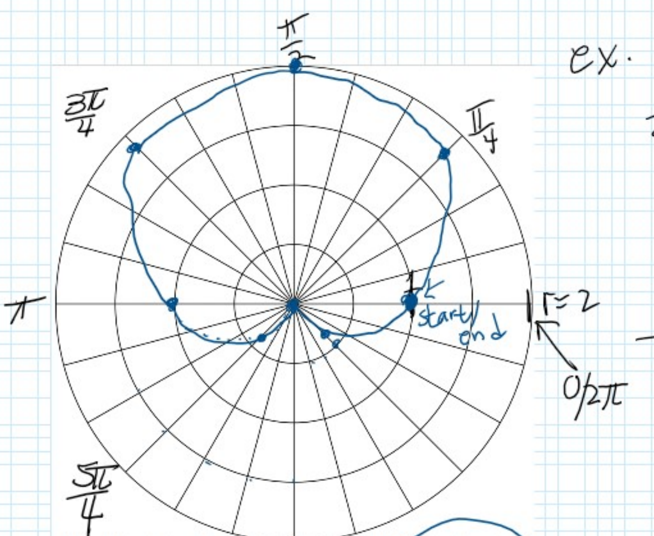
$x^2 + y^2 - 2x = 0$

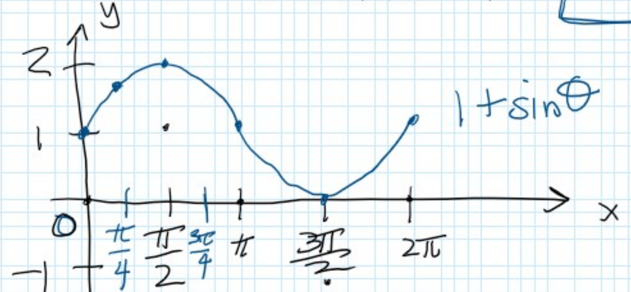
$\underbrace{x^2 - 2x + 1}_{(x-1)^2} + y^2 = 0 + 1$

$(x-1)^2 + y^2 = 1$

$(x-h)^2 + (y-k)^2 = r^2$

ex. sketch the curve $r = 1 + \sin\theta$



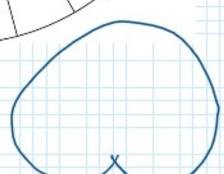


$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx .7 = r \Rightarrow 1 + \sin\frac{\pi}{4} = 1.7$

$1 + \sin\frac{3\pi}{4} = 1.7$

$1 + \sin\frac{5\pi}{4} = -\sin\frac{\pi}{4} + 1$

$= -\frac{\sqrt{2}}{2} + 1 \approx 1 - .7 = .3 = r$



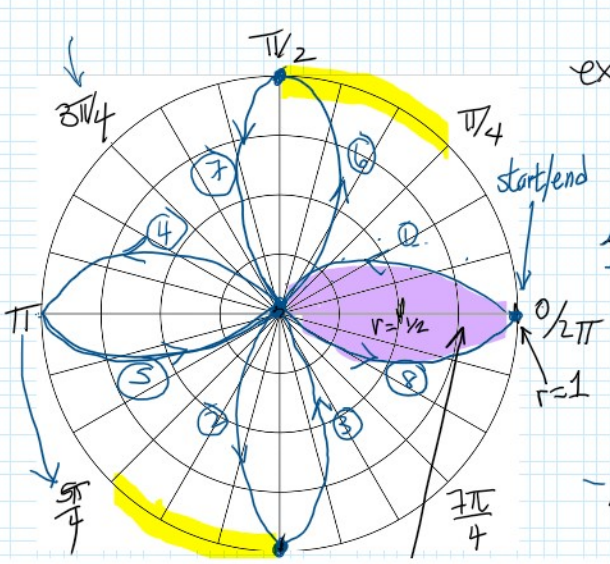
heart → CARDIOID

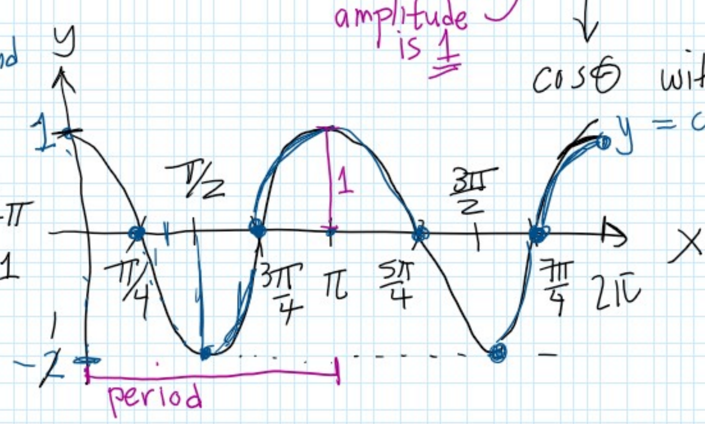
ex. sketch curve $r = \cos 2\theta$

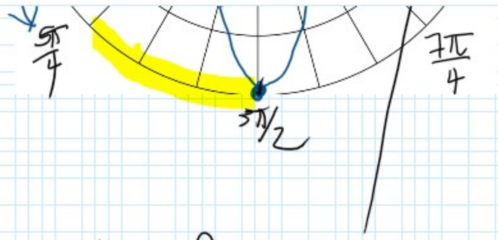
$\frac{2\pi}{2} = \frac{2\pi}{2} = \pi$

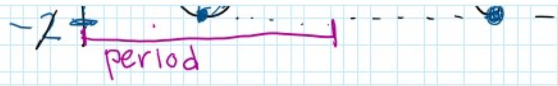
amplitude is 1

$\cos\theta$ with period π





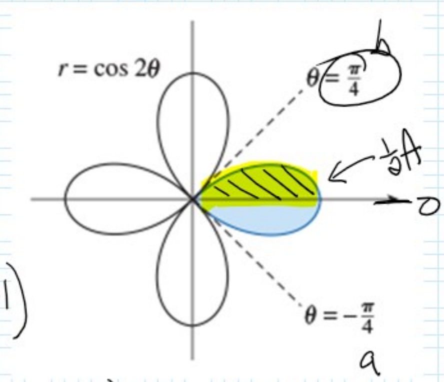




4-leaved rose

next: find area of one loop of $r = \cos 2\theta$

$$\boxed{A = \frac{1}{2}\int_a^b r^2 \, d\theta}$$



$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} (\cos 2\theta)^2 \, d\theta$$

$$A = \frac{1}{\cancel{2}} \cancel{2} \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

$\cos^2 \cancel{\theta} = \frac{1}{2}(\cos 2\theta + 1)$

2θ (2θ)

$$= \frac{1}{2}\int_0^{\pi/4} (\cos 4\theta + 1)\, d\theta$$

$= \frac{1}{2}(\cos 4\theta + 1)$

$$= \frac{1}{2}\left(\frac{1}{4}\sin 4\theta + \theta\right)\Big|_0^{\pi/4}$$

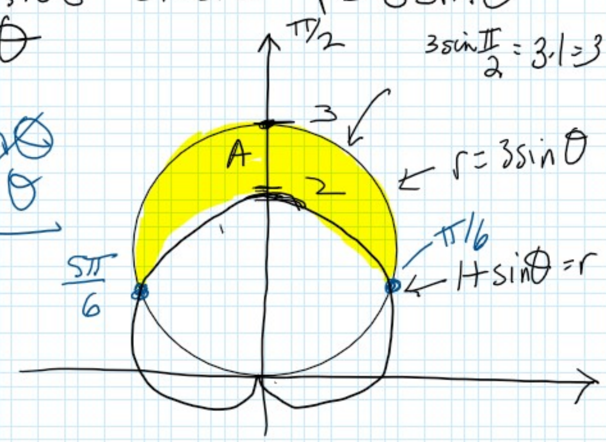
$$= \frac{1}{2}\left(\frac{1}{4}\left(\sin \cancel{4}\frac{\pi}{\cancel{4}} - \sin 0\right) + \frac{\pi}{4} - 0\right)$$

0 0

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$= \boxed{\frac{\pi}{8}}$$

ex. find area of the region that lies inside circle $r = 3\sin\theta$ and outside cardioid $r = 1 + \sin\theta$



find intersection points: $3\sin\theta = 1 + \sin\theta$

$-\sin\theta \quad -\sin\theta$

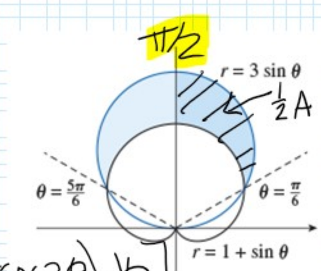
$2\sin\theta = 1$

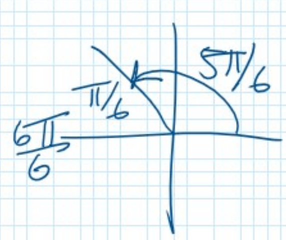
$\sin\theta = 1/2$

$\theta = \pi/6$

$$A = \frac{1}{2}\int_a^b r^2 \, d\theta$$

$$A = \frac{1}{2}\left[\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 \, d\theta - \int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2 \, d\theta\right]$$





$$= \frac{1}{2}\left[9\int_{\pi/6}^{5\pi/6} \sin^2\theta \, d\theta - \int_{\pi/6}^{5\pi/6} (1 + 2\sin\theta + \sin^2\theta)\, d\theta\right]$$

$$= \frac{1}{\cancel{2}}(\cancel{2})\left[+\frac{9}{2}\int_{\pi/6}^{\pi/2} (1 - \cos 2\theta)\, d\theta - \int_{\pi/6}^{\pi/2} \left(1 + 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right) d\theta\right]$$

etc.

$$= \boxed{\pi}$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$-\frac{1}{2}(\cos 2\theta - 1) = -\cancel{2}\sin^2\theta$

$\frac{1}{2}(1 - \cos 2\theta) = \sin^2\theta$